Seminararbeit zur ITYM – Teilnahme 2018

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Seminararbeit

Thema: ITYM – Teilnahme 2018

Zusätzlich Kurztitel (Nur falls das Thema mehr als 3 Zeilen zu je 44 Zeichen lang ist):

Verfasser/in: Elena Zeller

Leitfach: Physik

Seminarkürzel: 2PH_W

Lehrkraft: Herr Dr. Thomas Grillenbeck

Abgabe der schriftlichen Arbeit am: *[Datumsstempel Sekretariat]*

Präsentation mit Prüfungsgespräch am: _____._____. 20 _____

Bewertung	*Note*	*in Worten*	*Punkte*		*Punkte*
schriftliche Arbeit				*x 3*	
Abschlusspräsentation				*x 1*	
				Summe	
				Gesamtleistung** nach § 29 (7) GSO = **Summe : 2 (gerundet)	

Unterschrift der Lehrkraft

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1 Basics of the ITYM

1.1 What is the ITYM?

The ITYM is an international team contest for young mathematicians. Every year approximately 12 teams participate. These teams consist of high schoolers from all around the world. The first contest was hold in the year 2009; since then there has been a tournament every year. The teams in 2009s competition were from Belarus, Bulgaria, France and Russia. In the following year teams from Germany and Ukraine joined as well. Not all teams participated every year; however the following countries were all at least once represented in the competitions: Belarus, Bulgaria, France, Russia, Germany, Ukraine, Croatia, Romania, China, Iran, Nepal, Brazil, Georgia, India, Poland and Thailand. [11] This tournament has its roots in national tournaments, especially from eastern countries, and wants to "stimulate interest in mathematics and its applications, to develop scientific thinking, communication skills and teamwork". [2]

1.2 Basic regulatory for the tournament[3]

The language used in the contest is English, as the ITYM is an international contest. In the contest teams from different countries compete against each other. These teams consist of 4 to 6 high schoolers. Each team will be accompanied by one or two team leaders. The team leaders are supposed to supervise their protégées; however they are not supposed to help solving the problems. Every year there are about ten different problems the students are supposed to solve the best they can. These problems are published about 3 to 4 months before the competition starts. In this time the participating teams can prepare solutions to these problems. In the preparation time it is allowed to ask individual mathematicians for advice; however internet forums and other "public collective help" [4] are forbidden. The solutions have to be written down in the so called "written materials". The final written

[1] Cf. Homepage of the ITYM.
[2] Homepage of the ITYM.
[3] These regulations are mostly what I remember of the tournament. However all my knowledge of the regulatory is from the official regulatory. There will be a few additions and citations I took from the official regulatory of the ITYM; these will be marked as such.
[4] Regulatory of the ITYM.

materials have to be handed in four days before the tournament starts; after this deadline the solutions can't be changed.

The main part of the tournament consists of three rounds (Round 1, Round 2 and Finals). For each round the teams will be divided in groups of about 3 to 5 teams. In these groups each team will have to do a report on their solution for one of the problems. For this the Reporter has to explain the solution his team (for example: team 1) found for the corresponding problem. The Reporter is only allowed to use the content of the written materials and may only rectify minor mistakes.

This report has to be opposed by another team (in this example team 2). This will be done by an Opponent of team 2. The Opponent has to analyze the solution that team 1 gave in the written materials, as well as the presentation itself, and categorize it. [5] The Opponent is responsible to check for flaws in the solution and to start a debate with the Reporter of the other team about these aspects as well.

Next comes the role of the Reviewer; the Reviewer (of i.e. team 3) is supposed to sum up the report and the discussion between Reporter and Opponent and to evaluate whether the Opponent "said anything wrong or overlooked Reporter's faults".[6]

If there are more than 3 teams in one group the role of the Observer will be added. The Observer may only contribute to the discussion if Opponent and Reviewer missed an important mistake made by the Reporter or if he has anything else of importance to add to the discussion. However, if the Observer "wastes time, the Jury may evaluate the performance by negative marks." [5]

All of these roles will be awarded with points given by the jury. [7] The points of each team in the group will be divided by x. X is the amount of points the best team in the group reached. This amount of points will be the rating they get for this round. The best team in each group will have a rating of $\frac{x}{x} = 100\% = 1.0$ points for this round. The points of the two Rounds and the Quiz will determine which teams are going to participate in the grand and small finals. The finals determine the final ratings for the tournament. After the finals the scientific talks and award ceremony take place and with this the tournament ends.

[5] cf. Regulatory of the ITYM.
[6] Regulatory of the ITYM.
[7] The exact grading system can be seen in the Regulatory of the ITYM.

2 My time at the ITYM 2018

2.1 Preparation

My preparation for the tournament began when I was in class eighth. At this time I went with three other students of our school (all of them were in grade 10) to preparation meetings for the ITYM 2016 in St. Petersburg. This was kindly organized for us by our teacher Dr. Thomas Grillenbeck.

We met for two days in the “Schülerforschungszentrum” in Tuttlingen, one day in our school and travelled for one week to the “Hamburger Freiluftschule Wittenbergen” near Bremen. In these meetings we worked on solutions for the problems of the ITYM and were introduced to continuative mathematical thematics. In this year it was never planned that I participate in the tournament. Moreover it was a chance to learn about other mathematical fields and to get to know some of my future teammates and the supervisor, Mr. Helmut Ruf.

In the next year I participated in the tournament “Schüler experimentieren”. I worked together with another student in my grade on “Kruskal-Newton-Diagramme”. We won a second price for this.

When I was in grade 10, the year of the tournament, I solved parts of problem 5 at home. For this I was given a short introduction in graph theory by Peter Rottmeyr; he participated in the tournament a few years prior and is now studying mathematics at university. I sent my solutions to Mr. Helmut Ruf, as he organized the participation of team Germany 2 in the tournament. I was invited to the preparation meetings, because I showed effort and they already knew me from two years earlier.

For these meetings I travelled on two weekends to Tuttlingen. For me this meant a five hour trip by train to the “Schülerforschungszentrum” in Tuttlingen on Saturdays, as well as the five hours back home on Sundays. In these meetings we managed to solve some parts of the problems.

However we still wanted to improve our solutions. Therefore we met in Tuttlingen a week before the tournament and worked further on our solutions. We had to finish our solutions four days before the tournament began, and worked very hard to get the best results possible.

Our work didn't end after we handed our solutions in. The day after was the first draw; here the first opposing teams were announced (in our case it were France 2 and Georgia), as well as what problem we would have to report on first.[8] Two days before the tournament we got

Ill. 1: Our working space in Tuttlingen

the written solutions Georgia and France 2 had handed in for the problem they would report on. For these solutions we had to write so called "written reviews" that also had to be handed in before the first round. After drawing up the reviews, we had to leave for the tournament.

2.2 Experiences in the tournament

In my team were (from left to right in the picture) Elias Huber, David Ploss (team leader), Raphael Steiner, Noah Bihlmaier, Jonas Bear, Leon Duensing and I. The tournament started officially on July 6th 2018 in Paris in the ENS Jourdan University. Here we were welcomed by the organizers of the ITYM and sent to absolve the Quiz.

Ill. 2: Team Germany 2

In the Quiz we had to solve unknown problems in about 2.5 hours. These questions had similar subjects as the problems we had to solve in advance. In the Quiz each of us tried to solve the single questions by himself and only asked the others for help if needed. When the time was up, the Quizzes were graded while the opening ceremony and lunch took place. In the afternoon the first round started. We had to compete against France 2 and Georgia. I participated in this round as Reviewer for team Georgia on problem 10; however I couldn't really sum up much as the Reporter of Georgia only solved a very small part of the problem and the Opponent of team France could therefore not say much either. All in all we did very well in

[8] The exact process for the draw can be read in the official regulatory.

the first round; we got 92% of the points France 2 reached and got consequently 0.92 points.[9] In the evening we had the second draw in the hotel. Here we found out once again which teams we would have to compete against in the next round. The next day we had time to prepare the written reviews. The report and the opposition for the next round on July 8th had to be prepared as well. The second Round took place in the École des Ponts ParisTech, another Parisian university.

Ill. 3: École des Ponts ParisTech

We were in a group with team Germany 1 and team Russia 2.
This time we did even better than in the first round. We were awarded with the most points in our group from the jury. Hence we got 1.0 points. This result got us into the small final with France 1, Belarus 2 and Bulgaria. In the afternoon the Draw for the finals took place. July 10th we spent once again on preparing the written reviews, as well as the Report and the Opposition for the finals. The finals were held on July 11th in the headquarters of the Société Générale in Paris. We participated in the small final and reached 309 points; only one point short on team Belarus 2, the first in our group. All in all we reached the 6th place and were awarded with a third price together with Belarus 2 and France 1. The first place was achieved by team Belarus 2 followed by France 2, Romania and Russia 2. After the award ceremony, the scientific talks and the closing ceremony, the tournament had officially come to an end.

[9]All results can be seen under http://www.itym.org/results-2018

2.3 Solution to problem 5[10,11]

2.3.1 Short overview of the solution

In Problem 5 we examine the (b, α, d)-expansion in graphs.

First we proved that graphs exist for the given positive integers which meet the premises given in the problem, i.e. for the case where $\alpha = 2$ and $b = \frac{1}{2}$. The exemplary graph $G = (2, 1)$ fulfils all requirements for a (b, α, d)-expansion; for the case where $\alpha = 1, b = 1$ and $|V| \geq 2^d$ graphs n-dimensional cubes (Q_n) fulfil the premises, which we proved through induction.

In the next problem we had to prove for some special conditions that for a family of graphs G_n, $n \in \mathbb{N}$ there is, or is not, a positive α to get a valid (b, α, d)-expanse and find the greatest positive α, if such α exists: For $d = 2$ we can prove through contradiction that such α doesn't exist; for $d = 4$ we can prove that the maximum for α is given by $\alpha = \frac{1}{2}$, because else the graph wouldn't be (b, α)-weak expansion. By studying all different cases we can prove that the maximum for α is $\frac{1}{2}$.

Later we studied specific families of graphs, where we had to find the greatest positive α (or a prove that no such α exists), such that one can find a positive constant b, in order that this family is (b, α, d)-expander. At first, we had to investigate a family G_n, where G is a path with n vertices. We quickly found out that no such α fulfilling the conditions exists. After that we had to investigate a family G_n, where G is a (n^2)-rectangular grid on the Euclidean Plane, so $d = 4$. There, we at first claimed that such maximal α would be $\alpha = \frac{1}{2}$. After that, we proved that this really is the maximal possible value for α.

We provide solutions to problem 3 by giving the exact range of possible values α as desired. We do this by using a certain inequality which is a consequence of the expander lemma. Afterwards, we provide a complete solution to problem 4 by taking spectral estimates into account. Finally, we give a proof for the first claim of 5.5 using analytical arguments.

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[10] Problem 5 can be looked at in the attachments.

[11] This solution is the one Raphael Steiner and I wrote as our final written material in the tournament and was a combined team effort. It can still be looked at under the following link: https://drive.google.com/file/d/1CuicDyAGCV1qqOZ3COHjdYIeHO4vDZNS/view?usp=sharing; minor changes have been made.

2.3.2 Complete solution

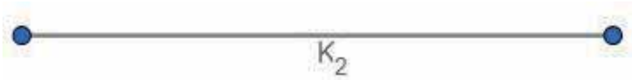
The graphs considered in the following are assumed to be simple.

Problem 5.1

For any fixed positive integer d construct a graph $G = (V, E)$, which admits (b, α, d)-expansion, where:

a) $\alpha = 2; b = \frac{1}{2}$.

Here, the:



Ill. 6: Graph K_2

admits $(\frac{1}{2}, 2, d)$-expansion.

In order for the graph to admit a $(\frac{1}{2}, 2)$-weak expansion, the equation $|\delta A| \geq b|A|^{\alpha}$ has to be valid. In case of the K_2, this is true, because $|A| = 1$, since otherwise $A = \emptyset$, which is forbidden. Thus $|A| \geq 1$. Since $|\frac{V}{2}| = 1 \rightarrow |A| \leq 1$. So $|A| = 1$, thus $|\delta A| \geq \frac{1}{2} * 1^2 = \frac{1}{2}$. Because the only edge in the graph is an edge between two vertices $v_i \in A$ and $v_j \in \overline{A}$, this proves the claim. The result is also the same for every $\alpha \in \mathbb{R}$ and for every $b \leq 1$, since $1^{\alpha} = 1; \forall\, \alpha \in \mathbb{R}$ and $1 * 1^{\alpha} \geq b * 1^{\alpha}; \forall\, b \leq 1$.

We furthermore notice that for each $d \in \mathbb{N}$, the d-regular complete graph K_{d+1} admits $(1, 2)$-weak expansion (and thus $(\frac{1}{2}, 2)$-expansion): For each $A \subseteq V(K_{d+1})$ with $|A| \leq \frac{d+1}{2}$, we have $|\delta A| = |A|\,|\overline{A}| \geq |A|^2$. Thus, the sequence $(K_{d+1})_{d \in \mathbb{R}}$ is (1,2)-expander.

b) $\alpha = 1; b = 1; |V| \geq 2^d$.

Since $|V|$ depends on d, one graph has to be constructed for every d. For $d = 1$, we have the same graph as in subtask 1, where $|1| \geq 1 * 1^1$. Thus the K_2 admits (1,1,1)-expansion. For $d = 2$, the graph of $d = 1$ gets expanded by one dimension. Now we have to fulfil the same conditions as above. Since A can be at most $\frac{|V|}{2} = 2$ and has to be at least 1: $A \in \{1,2\}$. For $|A| = 1$ the one vertex in A has always exactly two edges connecting it with two vertices in $\bar{A}$. Therefore $|\delta A| = 2$, which satisfies the equation $|\delta A| \geq b|A|^{\alpha} \rightarrow 2 \geq |A| = 1$.

For $|A| = 2$ whether two vertices in A are adjacent to each other, then $|\delta A| = 2$, or they are not, then $|\delta A| = 4$. Thus $|\delta A| \geq 2$. That means: $|\delta A| \geq b|A|^{\alpha} \rightarrow 2 \geq 2$.

Now we transfer this to an arbitrary d by induction. We want to show that for a $Q_d = (V_d, E.d)$ graph, for all $A < V_d$ with $|A| \leq \frac{|V_d|}{2} \rightarrow |\delta A| \geq |A|$.

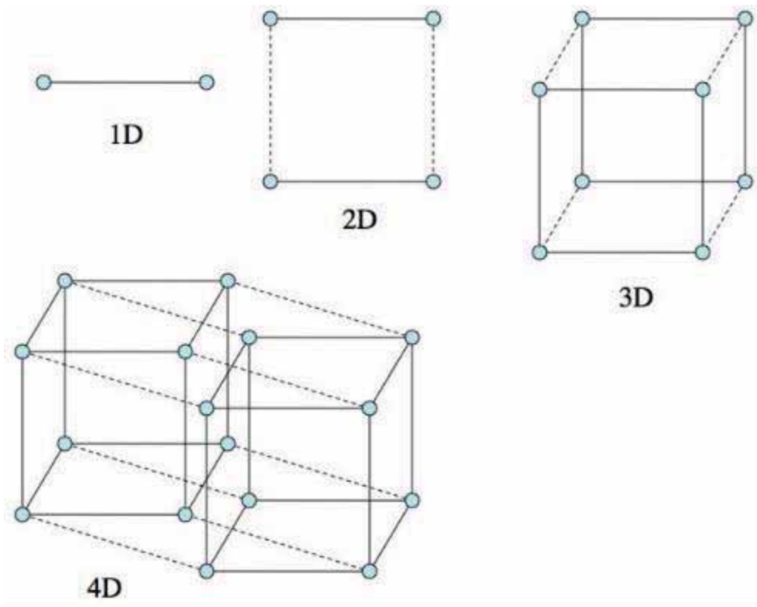


Illustration 7: The figure shows the evolution of d-dimensional hypercubes, from $d = 1$ to $d = 4$.

The start of the induction is given by the above for $d = 1$ and $d = 2$. Step of induction $d-1 \rightarrow d$(for every $d \geq 2$):

There have to be $A \subseteq V_d, |A| \leq \frac{|Vd|}{2} = 2^{d-1}$. We identify the vertices of G_d with (0,1)-vectors in the d-dimensional space. We may split A into disjoint subets A_1', A_2' which admit last coordinate 0 respectively 1. We denote A_1, A_2 their projections to the $(d-1)$-dimensional cube, i.e. they contain the $(d-1)$-dimensional 0,1-vectors obtained from the elements of A_1, A_2 by omitting the last coordinate. It is clear that $|A| = |A_1'| + |A_2'| = |A_1| + |A_2|$. They fulfil $|\delta A| = |A_1 \Delta A_2| + |\delta A_1| + |\delta A_2|$, because we either have an edge inside one of the "subhypercubes", meaning in δA_1 or δA_2 or a connection between those subhypercubes - such a connection is in δA

if $x \in A_1$, but not A_2 or the other way round the set is $A_1 \Delta A_2$. This can be adjusted, so the following equation is valid:

$|\delta A| = |A| - 2|A_1 \cap A_2| + |\delta A_1| + |\delta A_2|$. We show $|\delta A_1| + |\delta A_2| \geq 2|A_1 \cap A_2|$. We know: $|A| = |A_1| + |A_2| \geq 2^{d-1}$, so without loss of generality $|A_1| \leq 2^{(d-2)} = \frac{|V(G_{d-2})|}{2}$.

Then the following equations are valid: $|A_1| + |A_2| \geq 2|A_1 \cap A_2|$ and $|A_1| + |\bar{A}_2| = |A_1| + 2^{d-1} - |A_2| = 2|A_1| + 2^{d-1} - (|A_1| + |A_2|) \geq 2|A_1| \geq 2|A_1 \cap A_2|$.

Using the inductive assumption we have $|\delta A_1| + |\delta A_2| \geq |A_1| + min\{|A_2|, |\bar{A}_2|\} \geq 2|A_1 \cap A_2|$, as predicted in the original thesis.

Problem 5.2 a)

In this case, there is no positive value $\alpha > 0$ and $b > 0$ so that the sequence is (b, α)-expander. In order to see this, assume for contrary there was a pair of values (b, α), both positive, so that each G_n, i.e., the path on n vertices, is (b, α)-expanding. For even n, denote by $A_n \subseteq V(G_{2n})$ the set of the first n vertices on the path. Then obviously, $1 = |\delta A_n| \geq b|A_n|^\alpha = b \cdot n^\alpha, \forall n \in \mathbb{N}$ which is an obvious contradiction to $\alpha > 0$.

Problem 5.2 b)

We claim that such a maximal value is given by $\alpha = \frac{1}{2}$. First of all, the sequence is not (b, α)-expander for any $b > 0$ and $> \frac{1}{2}$: Given such a pair so that the sequence is (b, α)-expander, for each even $n \in \mathbb{N}$, take $A_n \subseteq V(G_{2n})$ to be the set of vertices on the $(2n)^2$ grid contained in the n leftmost columns. Then $|An| = 2n^2 \leq \frac{|V(G_{2n})|}{2}$, implying

$$(\mathbf{1})\ 2n = |\delta A_n| \geq b|A_n|^\alpha = b(2n^2)^\alpha = b2^\alpha n^{2\alpha}.$$

Since $n^{2\alpha-1}$ is unbounded for $\alpha > \frac{1}{2}$, this means that $\alpha \leq \frac{1}{2}$ for all expander-pairs (b, α).

On the other hand, we now show that (G_n) is $(\sqrt{2}, \frac{1}{2})$-expander, thereby proving the above assertion:

Let $A \subseteq V(G_n)$ with $|A| \leq \frac{|V(Gn)|}{2} = \frac{n^2}{2}$ be arbitrary. We need to show that $|\delta A| \geq \sqrt{2}|A|^{\frac{1}{2}} = \sqrt{2|A|}$, or equivalently $|\delta A|^2 \geq 2|A|$. For this purpose, we consider different cases:

(A) There is no row or column within G_n all of whose vertices are contained in A. If $A = \emptyset$, the claimed inequality obviously holds, so assume in the following $|A| \geq 1$. We will now conclude that

$$(\mathbf{2})\ |\delta A|^2 \geq 2\binom{|\delta A|}{2} \geq 2|A|.$$

For this purpose we construct an injective mapping $f: A \to \binom{\delta A}{2}$, where the latter denotes the set of 2-elements subsets of δA, proving the needed inequality $|A| \leq \binom{|\delta A|}{2}$:

For each $a \in A$, the column resp. row c_a, r_a containing a, according to the above, has to contain at least one element outside of A and thus at least one cut edge. We now define $f(a)$ to contain this pair of cut edges. f indeed is injective: Given the elements of $f(a)$, we may reconstruct the column resp. row a is contained in and thus the exact position of a itself.

(B) There is a row or a column fully contained in A. By rotational symmetry of G_n, we may assume without loss of generality that there is a designated full column c contained in A. We again distinguish two cases:

- There is no row fully contained in A. This means that in each row, we have at least one element not contained in A, and at least one element out of A (namely the crossing element with c). Thus, δA contains at least one edge within each column. We now conclude that (since $|A| \geq \frac{n^2}{2}$)

$$(\mathbf{3})\ |\delta A| \geq n = \sqrt{2}\sqrt{\frac{n^2}{2}} \geq \sqrt{2}\ |A|^{\frac{1}{2}}$$

  as claimed.
- There is a row r fully contained in A. Since each row or column crosses r or c, it contains an element of A. Let $k\ \leq 2n$ denote the number of rows or columns containing at least one element outside of A. In each such row or column, we may find a distinguished edge out of δA. Consequently, $|\delta A| \geq k$. On the other hand, $2n - k$ counts the number of rows/columns fully contained in A. Since $|A| \leq \frac{n^2}{2}$, there cannot be more than $\frac{n}{2}$ rows resp. columns

contained in A. Hence, we have $2n-k \leq \frac{n}{2}+\frac{n}{2}=n \Leftrightarrow k \geq n$. The latter now implies $|\delta A| \geq k \geq n$, and as above we conclude

$$(\mathbf{4})\ |\delta A| \geq n=\sqrt{2}\sqrt{\frac{n^2}{2}} \geq \sqrt{2}\ |A|^{\frac{1}{2}}.$$

Finally, since the claim was verified in each subcase, we conclude the claim.

Problem 5.3

We claim that the set of such α in the first case is given by the interval $(0,1]$. For this purpose, we first of all need to show that indeed for each $0<\alpha \leq 1$, there is a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs for some $d \in \mathbb{N}$ and $b>0$ which is (b,α,d)-expander. It obviously suffices to show this in the case $\alpha=1$. But then, such a sequence of 8-regular graphs indeed exists, cf. e.g. [12].

On the other hand, assume that contrary to the claim that there was a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs which is (b,α,d)-expander for some $b>0, d \in \mathbb{N}$. Then $|\delta A| \geq b|A|_\alpha, \forall\ A \subseteq V\ (G_n) : |A_n| \leq \frac{|V(Gn)|}{2}, n \in \mathbb{N}$. On the other hand, since for each $n \in \mathbb{N}, G_n$ is d-regular, the inequality given by the problem text yields that for all $A \subseteq V(G_n), |A| \leq \frac{|V(Gn)|}{2}$ and $n \geq 1$ we have

$$(\mathbf{5})\ b|A|^a \leq |\delta \mathrm{A}| \leq \frac{\mathrm{d}|\mathrm{A}||\bar{A}|}{|V(G_n)|}+d\sqrt{|\mathrm{A}||\bar{A}|}\ .$$

We may choose such sets $A_n \subseteq \mathrm{V}(G_n)$ for all $n \in \mathbb{N}$ of size $|\mathrm{A}_n|=\left\lfloor\frac{|(V(Gn)|}{2}\right\rfloor$. Putting this into the above inequality now yields (we use the notation $v_n := |\mathrm{V}\ (\mathrm{G}_n)|$):

$$(\mathbf{6})\ b\left\lfloor\frac{v_n}{2}\right\rfloor^a \leq \frac{d\left\lfloor\frac{v_n}{2}\right\rfloor\left\lceil\frac{v_n}{2}\right\rceil}{v_n}+d\sqrt{\left\lfloor\frac{v_n}{2}\right\rfloor\left\lceil\frac{v_n}{2}\right\rceil}\ \leq \frac{dv_n}{4}+\frac{dv_n}{2}=\frac{3dv_n}{4}.$$

Since $v_n \to \infty, n \to \infty$, for n big enough, we have $\left\lfloor\frac{v_n}{2}\right\rfloor \geq \frac{v_n}{4}$. Thus, we have (dividing by v_n):

$$(\mathbf{7})\ \frac{b}{4^\alpha} v_n^{\alpha-1} \leq \frac{3d}{4};\ (\alpha-1)>0$$

for n large enough, which is a contradiction (the left side grows arbitrarily for $n \to \infty$).

[12] Cf. Application of expander graphs.

In the second case, we consider the same question without the regularity restriction. We now claim that the set of admissible positive α is given by (0,2]. First of all, according to **1.**, $(K_{d+1})_{d\in\mathbb{N}}$ is (1,2)-expander and thus, indeed, each $0 < \alpha \leq 2$ is admissible. On the other hand, assume that contrary to the assertion, there is a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with $vn := |V(G_n)| \to \infty$, which is (b, α)-expander with $b > 0, \alpha > 2$. Then for each $n \in \mathbb{N}$, take some $A_n \subseteq V(G_n)$ with $|A_n| = \left\lfloor \frac{v_n}{2} \right\rfloor$. Then obviously, for n large enough,

$$(\mathbf{8})\ \frac{b}{4^\alpha} v_n^\alpha \leq b \left\lfloor \frac{v_n}{2} \right\rfloor^\alpha \leq |\delta A_n| \leq |A_n||\overline{A_n}| \leq \frac{v_n^2}{4}.$$

Dividing by v_n^2 implies

$$(\mathbf{9})\ \frac{b}{4^\alpha} v_n^{\alpha-2} \leq \frac{1}{4}$$

for n large enough. This is an obvious contradiction to $v_n \to \infty$, implying the second claim.

Problem 5.4

In [13], it is shown that for each $n \in \mathbb{N}$, the subgraph T_n' of the Cayley graph T_n which has the same vertex set and where two permutations s, t are adjacent if $s^{-1} \circ t$ is one of the transpositions $(1, n), (2, n), \dots, (n - 1, n)$, has minimal positive laplacian eigenvalue $\sigma_n' = 1$. According to the first inequality in the problem description (note that T_n' is $n - 1$-regular), we thus have $h(T_n') \geq \frac{\sigma_n'}{2} = \frac{1}{2}$. Hence, for all $A \subseteq V(T_n) = V(T_n'), |A| \leq \frac{|V(T_n)|}{2} = \frac{|V(T_n')|}{2}, |\delta_{T_n} A| \geq |\delta_{T_n'} A| = \frac{|\delta A|}{|A|}|A| \geq h(T_n')|A| \geq \frac{1}{2}|A|$

and consequently, $(T_n)_{n\in\mathbb{N}}$is $(\frac{1}{2}, 1)$-weak expander. This proves the claim.

Problem 5.5

We first prove that the given sequence is not $(b, 1)$-expander for some positive $b > 0$. For this purpose, we need the following result from [14]: For every $n \in \mathbb{N}$, the minimal positive laplacian eigenvalue of B_n is given by $\sigma_n = 2\left(1 - cos\left(\frac{\pi}{n}\right)\right)$. Since for each n, B_n is $(n - 1)$-regular, we may apply the first inequality in the problem description and deduce

$$(\mathbf{10})\ h(B_n) \leq \sqrt{2\sigma_n(n-1)} \leq \sqrt{2\sigma_n n}.$$

[13] Cf. Cayley Graphs.

[14] Cf. Laplacien de Coxeter.

We now prove that $\sigma_n n \to 0, n \to \infty$. For this purpose, we use Taylor's theorem and the fact that $|\sin|, |\cos| \leq 1$ in order to deduce

$$(\mathbf{11})\ |\cos(x) - \left(1 - \frac{x^2}{2}\right)| \leq \frac{1}{6}|x|^3$$

for all $x \in \mathbb{R}$. Applying this to the above yields

$$(\mathbf{12})\ \sigma n = |\sigma n| = 2\left|1 - \frac{\pi^2}{2n^2} - \cos\left(\frac{\pi}{n} + \frac{\pi^2}{2n^2}\right)\right|$$

$$\leq 2\left|\cos\left(\frac{\pi}{n}\right) - \left(1 - \frac{\left(\frac{\pi}{2}\right)^2}{2}\right) + \frac{\pi^2}{n^2}\right|$$

$$\leq \frac{1}{6}\frac{\pi^3}{n^3} + \frac{\pi^2}{n^2} \leq \frac{C}{n^2},$$

where $C \in \mathbb{R}_+$ is a positive constant independent of n.

This finally implies $0 \leq \sigma_n n \leq \frac{C}{n} \to 0$ as claimed.

Now assume that contrary to the assertion there was a $b > 0$ so that (B_n) is $(b, 1)$-expander.

This means that $|\delta A| \geq b|A|$ for all $A \subseteq V(Bn), |A| \leq \frac{|V(B_n)|}{2}$ in B_n and for all $n \in \mathbb{N}$, i.e., $h(B_n) \geq b, n \in \mathbb{N}$. Finally, we have

$$(\mathbf{13})\ 0 < b \leq h(B_n) \leq \sqrt{2\sigma_n n} \to 0;\ (n \to \infty)$$

which is the desired contradiction.

2.4 Experiences outside of the tournament

The best thing about this tournament were the teams from all around the world and the resulting internationality. As I was the only girl in our group I had a room with five other girls. Two of them were French and the other three were from the Indian team. I feel very lucky to have been placed in a room with them as all girls were very friendly. Especially the Indian girls were very open and welcoming. We often talked and through this I got to know them and their culture very well. I got to know students from other teams as well. The many social events or the time during the breaks were good opportunities to get in contact with other teams. And, as during the time of the tournament the soccer world cup took place in Russia, it was always easy to start conversations (either about the outcome of the latest game or predictions for the next match). In some of the social events we watched the semifinals of the world cup. And this was not the only time people watched the world cup: In the break of the first round all French team members gathered around a mobile phone with the broadcast of the match France against Uruguay. The next day some jury members met to watch the game "England against Switzerland" on a beamer; in the meantime we were working on our written materials for the next round.

Another highlight of the contest was the sightseeing day that was scheduled for July 9th. We first took the train to the Louvre. Here we were given free time to explore the museum and to have lunch in the park next to the Louvre.

Ill. 4: The Louvre

After this we visited Montmartre. It's one of Paris' best known urban districts. We ate Japanese Ice Cream, visited different churches, walked through small alleyways and enjoyed the magnificent view of the whole city.

Ill. 5: Team Germany 2 in Montmartre

For the next stop we visited the Eiffel tower. Last but not least we did a boat trip on the Seine where we saw Notre Dame and different Parisian urban districts before we returned to the hotel.

Concluding I have to say that the tournament was a great experience I wouldn't want to miss.

3 Acknowledgements

I would like to thank quite a few people and institutions:

- My teacher Dr. Thomas Grillenbeck. He offered me the opportunity in eight grade to join the preparation meetings and always helped me if I had any questions.
- Mr. Helmut Ruf and Prof. Dr. Dierk Schleicher for organizing the participation of our team in the tournament and the preparation meeting in Bremen.
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- Peter Rottmeyr for giving me an introduction in graph theory.
- The "Kultur- und Sozialstiftung des Oberbürgermeisters der Stadt Rosenheim Dr. Michael Stöcker". They sponsored my trip to Paris with 500€.
- The Schülerforschungszentrum in Tuttlingen. They provided the accommodation and meals for the preparation meetings.

They made my participation in this tournament possible. Therefore I would like to express my sincere thanks to all parties involved once again.

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4.1 Literature

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- Regulatory of the ITYM (https://docs.google.com/viewer?a=v&pid=sites&srcid=aXR5bS5vcmd8d3d3fGd4OjdjZGVmNmRlZTIyYTliZTg)
- Results of the ITYM 2018 (http://www.itym.org/results-2018)
- Elena Zeller, Raphael Steiner on Problem 5: Expansion in Graphs (https://drive.google.com/file/d/1CuicDyAGCV1qqOZ3COHjdYIeHO4vDZNS/view?usp=sharing)
- Problem 5 (https://docs.google.com/viewer?a=v&pid=sites&srcid=aXR5bS5vcmd8d3d3fGd4OjNjYjYyNWIwYmMwYWZkNjQ)

5 List of illustrations

6 Attachments

Problem 5 and my certificate of the tournament can be seen on the next pages.

5. **Expansion in Graphs**

Let b, d, α be positive constants. We consider a connected undirected graph $G = (V, E)$, where V and E are the sets of vertices and edges respectively. For any nonempty proper subset $A \subset V$ denote by ∂A the set of edges $(v_i, v_j) \in E$ with $v_i \in A$, $v_j \in \overline{A} = V \setminus A$. We say that the graph G admits (b, α)-*weak expansion* if $|\partial A| \geq b|A|^\alpha$ for any nonempty proper subset $A \subset V$ with $|A| \leq |V|/2$. If the graph G admits (b, α)-weak expansion and any vertex from V has degree at most d, we say that the graph G admits (b, α, d)-*expansion*. Graph G is called d-regular if degree of any vertex of G is equal to d. Suppose there is given a sequence of connected undirected graphs $G_n = (V_n, E_n)$ such that $|V_n| \to \infty$ as $n \to \infty$. We say that the sequence G_n is (b, α)-*weak expander* if any graph G_n admits (b, α)-weak expansion. Analogously we say that the sequence G_n is (b, α, d)-*expander* if any graph G_n admits (b, α, d)-expansion.

Below are the main tools from theory of expanders. To learn more about the topic, one can also try *Expander Families and Cayley Graphs: A Beginner's Guide* by M.Krebs and A.Shaheen.

The Cheeger constant is defined as

$$h(G) = \min_{A \subset V, 0 < |A| \leq \frac{|V|}{2}} \frac{|\partial A|}{|A|}.$$

The spectral gap σ of a graph G is the smallest positive eigenvalue of its Laplacian matrix.

Let G be a connected undirected d-regular graph, then the Cheeger inequality states

$$\frac{\sigma}{2} \leq h(G) \leq \sqrt{2\sigma d}.$$

Let G be a connected undirected d-regular graph on n vertices. Corollary from the expander mixing lemma gives the inequality

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Let G be a connected undirected d-regular graph on n vertices. Corollary from the expander mixing lemma gives the inequality

$$\left| |\partial A| - \frac{d|A||\overline{A}|}{n} \right| \leq d\sqrt{|A||\overline{A}|}.$$

1. For any fixed positive integer d construct a graph $G = (V, E)$, which admits (b, α, d)-expansion, where:

a) $\alpha = 2$, $b = 1/2$;

b) $\alpha = 1$, $b = 1$, $|V| \geq 2^d$.

2. Find the greatest positive α (or show that such α does not exist) such that one can find positive constant b such that family of graphs G_n, $n \in \mathbb{N}$, is (b, α, d)-expander, where:

a) G_n is the path on n vertices ($d = 2$);

b) G_n is the graph of the $(n \times n)$-rectangular grid on the Euclidean plane ($d = 4$);

c) G_n is the graph of the $(n \times n \times n)$-grid in the three-dimensional Euclidean space ($d = 6$).

3. Find the set of positive α such that one can find positive b, d and a family of d-regular graphs G_n, $n \in \mathbb{N}$, which is (b, α, d)-expander. Investigate the same problem for the graphs, which are not d-regular in general case.

4. Let's consider the set S_n of all permutations of the set $\{1, \ldots, n\}$. Define graph T_n as follows: the vertices of T_n are the elements of S_n, any two vertices are connected by an edge if and only if for the corresponding permutations s, t the element $s^{-1} \circ t$ is a transposition. Prove that the sequence T_n is $(1/2, 1)$-weak expander.

5. Construct the graph B_n by using the same rules as in the previous question, but getting only transpositions $(i-1, i)$, $i = 1, \ldots, n$, instead of all transpositions. Show that the sequence B_n is not $(b, 1)$-weak expander for any positive b. Find the largest $\alpha > 0$ such that for any $\alpha_1 < \alpha$ one can find $b_1 > 0$ such that the sequence B_n is (b_1, α_1)-weak expander.

6. Suggest and investigate your own directions of this problem.

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